Calcul détaillé d'une Inégalité Nouvelle à Longue Période, qui existe dans la Longitude moyenne de la Lune. By M. Hansen.

The author states that he has lately made known to some astronomers a discovery of two inequalities in the motion of the moon, whose periods are respectively nearly 273 and 239 years. Denoting by g, g', g'', the geocentric mean anomaly of the moon, and the heliocentric mean anomalies of the earth and *Venus*, these inequalities are,—

$$27''\cdot4 \times \sin.\ (-g - 16\,g' + 18\,g'' + 35^\circ\ 20'\cdot2)$$
$$+\,23''\cdot2 \times \sin.\ (8\,g'' - 13\,g' + 315^\circ\ 30');$$

of which the first depends on a new argument, while the second depends on the argument of an equation of long period in the motion of the earth, discovered by Mr. Airy.

As the calculation of those parts of the coefficients which depend on the product of the square and cube of the sun's disturbing force by the disturbing force of *Venus* is extremely laborious, and is, moreover, connected with other unpublished calculations of other inequalities of the moon, it does not appear possible to publish it at present. Indeed Mr. Hansen does not consider himself able yet to answer for their perfect correctness, though he has the strongest reason to believe that they are very nearly correct. The present paper, therefore, includes only the calculation of that part of the coefficient of the first inequality which depends on the first power of the disturbing force.

It appears difficult to abstract very completely the remainder of this paper, but the following indications will enable a person acquainted with the developements of physical astronomy to follow the whole process.

The perturbing function Ω for the moon as disturbed by *Venus* being formed, it will be found that it may be expanded in a rapidly converging series of fractions, whose numerators contain successive powers of r, the moon's radius vector, and whose denominators contain different powers of the same multinomial (which, when eccentricities and inclinations are omitted, is a trinomial) that occurs in computing the perturbations of the earth by *Venus*. Upon expanding any of these fractions with trinomial denominator, there occur terms depending on $16\,g'' - 16\,g'$, $17\,g'' - 17\,g'$, and $18\,g'' - 18\,g'$: then, upon introducing the inclinations and eccentricities, the first (among other combinations) will be multiplied by $\sin.^2 \frac{1}{2}$ inclin. $\times \cos.\ \overline{2\,g'' - 2\nu}$ (where ν is the difference of longitude of node and perihelion of *Venus*), and also (in other terms) by $e''^2 \cos.\ 2\ g''$; the second by $e''.e' \cos.\ \overline{g'' + g'}$; and the third by $e'^2.\cos.\ 2\ g'$. Each of these combinations produces terms whose argument is $18\,g'' - 16\ g'$. Then upon multiplying these terms by a power of r, since the expression for any power of r contains $e.\cos.\ g$, the product will contain terms depending on $18\,g'' - 16\,g' - g$.

The coefficient necessarily contains one of the following products of three small quantities: $e.\sin.^2 \frac{1}{2}$ inclin., $e.e''^2$, $e.e''.e'$, $e.e'^2$ (of which the first is the most important), and it is, therefore, extremely small; but the resulting perturbation is made important by the excessive smallness of the divisor introduced in integration. It is well known that the divisor in this case will be proportional to $(18\frac{dg''}{dt} - 16\frac{dg'}{dt} - \frac{dg}{dt})^2$; and, taking for $\frac{dg''}{dt}$, &c., the value in sexagesimal seconds corresponding to a Julian year,

$$\frac{dg''}{dt} = 2106641''\cdot 3$$

$$\frac{dg'}{dt} = 1295977\cdot 4$$

$$\frac{dg}{dt} = 17179157\cdot 4$$

whence $18\frac{dg''}{dt} - 16\frac{dg'}{dt} - \frac{dg}{dt} = 4747''\cdot 7$,

a quantity very small in comparison with $\frac{dg}{dt}$.

In this manner the greatest part of the term in question is produced. Other parts arise from the circumstance that, the dimensions of the moon's orbit being slightly altered, the perturbing force of the sun upon the moon is not the same as it would otherwise be.

M. Hansen remarks that this term is remarkable as depending upon higher multiples of the anomalies than have ever before been considered, and as having the longest period in proportion to the periodic time of the disturbed body that is yet known.

The term depending on $8g'' - 13g'$ arises mainly from the circumstance that, the earth's motion in its orbit being different from what it would have been without the perturbation by *Venus*, the disturbing force of the sun upon the moon is not the same as if that perturbation had not existed.

M. Hansen states that he has examined several inequalities of long period in the moon's motion which hitherto have escaped notice, but that in no other instance does the coefficient amount to 1''.

In concluding the account of this remarkable discovery, it is gratifying to add that it explains almost precisely the observed inequality in the moon's mean motion, which, for the last fifty years, has troubled physical astronomers.

After the reading by the Secretary of a portion of this paper, the Astronomer Royal gave an oral explanation of its general subject in the following manner:—

The disturbing effect of *Venus* upon the moon, is not the whole attraction of *Venus* upon the moon, but the difference of the two attractions, of *Venus* upon the moon and of *Venus* upon the earth. Thus, when the moon is between the earth and *Venus*, the attrac-

tion of *Venus* upon the moon is stronger than that of *Venus* upon the earth, and, therefore, it tends to pull *Venus* away from the earth. When the moon is more distant from *Venus* than the earth is, the attraction of *Venus* on the earth is the stronger, and tends to pull it away from the moon, which, in regard to the disturbance of the relative places of the earth and moon, is the same thing as pulling the moon away from the earth. In both these positions, therefore, the disturbing force of *Venus* tends to pull the moon away *from* the earth. When the earth and the moon are equally distant from *Venus*, the attractions of *Venus* upon the two are equal, but not in parallel lines; the attractions tend to draw them along the sides of a wedge whose point is at *Venus*, and, therefore, to diminish the distance between them, or to push the moon *towards* the earth.

Inasmuch as, in one pair of positions of the earth and moon, the disturbing force of *Venus* tends to increase the distance between them, and in another pair of positions it tends to diminish that distance, it is important to ascertain which of these disturbances is the greater. Suppose the distance of the moon from the earth to be $\frac{1}{100}$ part of the distance of the earth from *Venus*. Then, when the moon is between the earth and *Venus*, its distance from *Venus* is $\frac{99}{100}$ of the whole; the force upon it is $\frac{10000}{9801}$ of that upon the earth; the excess of this (or the disturbing force tending to pull the moon away from the earth) is $\frac{199}{9801}$, or nearly $\frac{1}{50}$ of that on the earth. In like manner, when the moon is further from *Venus* than the earth is, its distance from *Venus* is $\frac{101}{100}$ of the earth's distance; the force upon it is $\frac{10000}{10201}$ of that upon the earth; the defect of this (or the disturbing force tending to pull the earth away from the moon) is $\frac{201}{10201}$, or nearly $\frac{1}{51}$ of that on the earth. But when the earth and the moon are at equal distances from *Venus*, the proportion of their relative approach (as produced by the action of *Venus*) to the whole effect of *Venus* upon them, is evidently represented by the inclination of the two lines drawn from them to *Venus*, or is the same as the proportion of the distance of the moon from the earth, to the distance of the earth from *Venus*, and is, therefore, $\frac{1}{100}$ of the whole. Thus, the force tending to pull the moon from the earth at one time is about double the force tending to push the moon towards the earth at another time; and, therefore, upon the whole, the tendency of the disturbing force of *Venus* is to pull the moon from the earth. To arrive at this conclusion, we have considered only four points of the moon's orbit: in other points the effects of the perturbation are more complicated; but they do not alter this general conclusion.

The same remark applies to the disturbing effect of *Venus* upon the moon when at a given point of its orbit, provided the nature of that point be such that at different times it is in all possible positions relative to *Venus*. For instance, the moon's apogee is (in consequence of the motion of the line of apses, and of the relative motions of the earth and *Venus*) sometimes between the earth and *Venus*, sometimes more distant from *Venus* than the earth is, some-

times 90° to the right, sometimes 90° to the left. We may assert, therefore, that, upon the whole, the disturbing force of *Venus* upon the moon, when she is in apogee, tends to draw her away from the earth. The same may be predicated when the moon is in perigee.

Next, it is important to ascertain how the disturbing force depends upon the moon's distance from the earth. For this purpose, instead of supposing, as before, that the moon's distance is $\frac{1}{100}$ part of the distance of the earth from *Venus*, let us suppose it $\frac{2}{100}$ part of that distance. Then when the moon is between the earth and *Venus*, the force upon the moon is $\frac{10000}{9604}$ of that upon the earth, and, therefore, the excess, or the disturbing force, is $\frac{396}{9604}$, or nearly $\frac{1}{25}$ of the whole force upon the earth. In the former assumed instance it was $\frac{1}{50}$. Thus, upon doubling the moon's distance from the earth, the disturbing force is doubled. And similarly for other distances of the moon from the earth, the disturbing force (in similar positions with regard to *Venus*) is proportional to the moon's distance. Thus, when the moon is at apogee, in a given position with regard to *Venus*, the disturbing force is greater than when the moon is in perigee in the same position. And, upon the whole, in all possible relative positions of the moon and *Venus*, the action of *Venus* pulls away the moon from the earth, more when she is in apogee than when she is in perigee.

Now we may consider the general effect of these forces upon the dimensions of the moon's orbit. So long as the force which draws the moon towards the earth is always the same at the same distance, the moon will continue to describe an orbit of the same dimensions over and over again. But if at any time the force directed towards the earth *suddenly* grows smaller, the moon will *immediately* rush off in an orbit which, on the opposite side, is larger. If the force towards the earth *gradually* grows smaller, the dimensions of the orbit will *gradually* increase. And the periodic time in the orbit described at every successive revolution will undergo the change corresponding to the change of dimensions (that is, to the change of major axis) of the orbit, and will, therefore, become continually greater and greater.

These are the changes which produce the most serious disturbance in the apparent place of the moon. If a force, after acting for a long time, produce a small change in the eccentricity of the moon's orbit, the effect on the moon's place is simply the amount of the corresponding change in the equation of the centre, and cannot possibly exceed that amount. But if the force have been for a long time gradually altering the major axis, and, consequently, the periodic time in the moon's orbit, then during the whole of that time the moon has been performing her revolutions quicker or slower than we expected, and, therefore, at the end of that time she is in advance or in retard of her expected place by an amount equal to the accumulation of all the advances or retards in all the revolutions through which the change has been going on. The planetary inequalities of long period are all of this kind. The major axis here plays the same part as the pendulum of a clock: if

a small force acting for a year pushed the seconds-hand forwards by an inch, the clock would be merely a few seconds wrong; but if, in the same time, it shortened the pendulum by an inch, the clock would have gained 50 hours; and if the time occupied by the change had been greater, the disturbance in the clock indication would have been proportionably greater.

In order, then, to find inequalities of long period in the motion of the moon produced by *Venus*, we must seek for some alternate increase and decrease, occupying a very long period, in the force by which *Venus* draws the moon from the earth.

No such slow increase and decrease have been found in the general force by which *Venus* disturbs the moon.

The next point of inquiry is, whether a combination of the changes in the force of *Venus* with the changes in the position of the moon in its orbit can produce a force which, for a very long time together, gradually increases the force drawing the moon from the earth, and then for an equal time gradually diminishes that force.

A force which acts in opposite ways, nearly on opposite sides of the moon's orbit (pulling the moon from the earth on one side and pushing it towards the earth on the other side), may produce this effect, provided the period of the change in the nature of the force (from pulling to pushing) correspond *nearly*, but *not exactly*, with the time in which the moon moves from apogee to perigee. For (as we have seen) the effect of a certain force of *Venus* is to produce a greater disturbing force on the moon at apogee than at perigee; and this force, or a change in this force, will, at apogee, produce a greater effect on the dimensions of the moon's orbit than at perigee, both because the disturbing force is actually greater, and because it acts on the moon when the moon's velocity is smaller. Therefore, if a pulling force, gradually increasing in magnitude, act on the moon at apogee, it will gradually increase the dimensions of the moon's orbit; if a corresponding pushing force act at perigee, it will gradually diminish the dimensions of the moon's orbit; but the former prevails, and the orbit will gradually increase in size. If, after a time, the pulling force at apogee gradually diminish, and at length become a pushing force, while the pushing force at perigee gradually diminishes, and at length becomes a pulling force, then the orbit will gradually diminish in size. And this change of forces would be produced by such a modification in *Venus's* force, as that of which we have spoken, namely, a force which acts in opposite ways on opposite sides of the moon's orbit, and in which the period in the change of the nature of the force coincides *nearly*, but *not exactly*, with the time in which the moon moves from apogee to perigee; for then the pulling force at apogee will, after a long time, be changed to a pushing force, and the pushing force at perigee will, in the same time, be changed to a pulling force. If, for instance, the change in the disturbing forces of *Venus* (from pushing to pulling) occupied 14 days exactly, and if the moon's motion from apogee to perigee occupied 14 days and 5 minutes, then in 4032 anoma-

listic semi-revolutions of the moon (which would bring her from apogee to apogee), there would have been 4033 changes of the force (which would change it from pulling to pushing), and, therefore, in this time, and no sooner, a complete pulling force at apogee would be changed to a complete pushing force at apogee.

It is necessary now to point out how such a modification of the force of *Venus* can be found.

The only disturbing forces which are yet completely brought under the management of mathematicians are of two kinds:—a constant force (always pushing or always pulling with the same amount of force); and a force alternately pushing and pulling, having equal periods and equal maximum magnitudes in each state. The latter of these, if projected graphically, with the time for abscissa, is represented by the ordinates of a *line of sines*: algebraically, it is expressed by $a.\cos(b\,t + c)$.

Now, while the relative positions of the earth and *Venus* change, the disturbing force on the moon (estimated by the force which, on the whole, it exerts to pull the moon from the earth) undergoes very great changes. When *Venus* is nearest to the earth, this force is about 250 times as great as when *Venus* is furthest from the earth. It declines very rapidly from its greatest magnitude. If, therefore, we represent the disturbing force from one conjunction to the next, by a curve, this curve will be very high at the beginning and end, and very near the line of abscissa at the middle, and through the greater part of its extent.

The separation of this force into a number of different forces, following the two laws mentioned above, is effected by a process suggested and facilitated by algebra, but in which, nevertheless, every step has its physical meaning. It may be stated, at once, that this remark applies universally to the algebraical operations of physical mathematics. As a simple instance, we may refer to the equation $(a+b)^3 = a^3 + 3a^2 b + 3 a b^2 + b^3$, which probably was suggested by algebra; but which may be illustrated by taking a cube, whose side is $a+b$, and (by three saw-cuts) cutting it into eight pieces, when the single piece representing a^3, the three pieces each representing $a^2 b$, the three pieces each representing $a b^2$, and the single piece representing b^3, will be found. And there is, perhaps, no better discipline for the mind than thus tracing the evidence of the truth of algebra, especially in its more profound processes.

The separation, then, of the force of *Venus* goes on by the following steps:

1st. A constant pulling force, equal to the mean value of the force.

2d. A force pulling when *Venus* is in conjunction, pushing at the time intermediate to the conjunctions, and pulling when *Venus* is in conjunction again: thus going completely through its changes *once* between conjunction and conjunction.

3d. A force pulling when *Venus* is in conjunction, then pushing, &c., going through its changes *twice*.

4th. A force pulling when *Venus* is in conjunction, then pushing, &c. going through its changes *thrice*.

In this manner the forces go on, continually diminishing in magnitude. When we arrive at the 18th, the force is extremely small.

The algebraical expression for the collection of these terms, putting θ for the difference of mean longitude of the earth and *Venus*, is

$$A + B \cdot \cos\theta + C \cdot \cos 2\theta + D \cdot \cos 3\theta + \&c.$$

This is on the supposition that the orbits of the two planets are circular and in the same plane. But, in consequence of their eccentricities and inclinations, the forces of any one system alternately pushing and pulling (Nos. 2, or 3, or 4, &c.) will not have the same *maximum* magnitude throughout. But each can, in all cases, be expressed by the combination of three such forces, in each of which the maximum forces are equal throughout. Thus, if we combine a large force, going through its changes 20 times in a certain period, with a small force, going through its changes 19 times in the same period, and another small force, going through its changes 21 times in the same period, then it will be found that both the small forces increase the large force (whether in its pulling or in its pushing state) near the beginning and the end of the time; that both diminish the large force near the middle of the time; and that the two small ones destroy each other at $\frac{1}{4}$ and $\frac{3}{4}$ of the time. The effect of this combination is, therefore, precisely such as is spoken of above.

Thus, then, for the complete expression of the force, we are driven to an infinite number of forces following the law of alternately pulling and pushing, but with very great variety of magnitudes of force and of periodic time. The greatest portion of these produce no sensible effect; some because (though their magnitudes are large) they act for so short time in one way, or their periods are so little related to the periods of any movement of the moon, that their effects never accumulate; others because their magnitudes are small, and there is no unusual circumstance favourable to their increase.

But there is one of these forces which, in the algebraical expression, depends on 18 × mean longitude of *Venus* — 16 × mean longitude of the earth, whose coefficient is exceedingly small, but which goes through all its changes, from pulling to pulling again, in the time,

$$27^{d}\ 13^{h}\ 7^{m}\ 35^{s}\!\cdot\!6;$$

or from pulling to pushing, in the time

$$13^{d}\ 18^{h}\ 33^{m}\ 47^{s}\!\cdot\!8.$$

Now, the anomalistic revolution of the moon, from apogee to apogee again, is performed in the time

$$27^{d}\ 13^{h}\ 18^{m}\ 32^{s}\!\cdot\!3;$$

or from apogee to perigee, in the time

$$13^{d}\ 18^{h}\ 39^{m}\ 16^{s}\cdot 1.$$

Here we have a real instance, exactly corresponding to the case which, for the sake of explanation, we assumed a short time back, and the results are truly such as were there described. During about 4000 half-revolutions of the moon, or 2000 revolutions, the pulling force at apogee is gradually diminishing till it becomes a pushing force, and during about 2000 more revolutions, the pushing force at apogee is gradually diminishing till it becomes again a pulling force; the opposite changes going on in the force at perigee: and thus, for reasons fully explained before, the moon's orbit is gradually contracting during 2000 revolutions, and gradually expanding during 2000 revolutions more. And although the change in the size of the orbit is totally insensible in observation (for, according to a rough calculation, the utmost accumulation of change in the major axis of the moon's orbit is only 10 *feet*, sometimes in increase and sometimes in decrease), yet the consequent alteration in its periodic time, continued through so many revolutions, is sufficient to cause the irregularity in question. The inequality in longitude, as measured on the moon's orbit, exceeds 30 *miles*, sometimes in advance and sometimes in retard.

For a complete understanding of this matter, it must carefully be borne in mind, that the force at the apogee, which has been described as a pushing force through 136 years, is not absolutely a pushing force through every month of that time, but that (in consequence of the motion of the moon's line of apses), if we take any period of 9 or 10 years, the moon's apogee will, in that time, have passed through every position with regard to *Venus*, and, therefore, *upon the whole*, during that period of 9 or 10 years, the force at apogee will have been a pushing force. In like manner, in another period of 136 years, if we take any period of 9 or 10 years, *upon the whole*, during that period of 9 or 10 years, the force at apogee will have been a pulling force.

The general cause of the inequality depending on the argument $8g'' - 13g'$, has been sufficiently stated in one of the last paragraphs of the Abstract of M. Hansen's paper.

After the Astronomer Royal had finished his lecture, the thanks of the Society were voted to M. Hansen for the communication of his Memoir, and to the Astronomer Royal for his lucid and popular explanation of its significance and bearing. The President briefly expressed his feeling of the honour which M. Hansen had done the Astronomical Society, by making it the channel for publishing his profound and brilliant researches, and expressed his admiration of the manner in which Mr. Airy had brought the results of the most refined and delicate analysis within the scope of ordinary comprehension.

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